Journal of Statistical Physics, Vol. 77, Nos. 1/2, 1994

A Goldstone Mode in the Kawasaki–Ising Model

Claudio Albanese[1, 2]

Received June 7, 1993; final April 29, 1994

The hydrodynamic regime of superfluids is dominated by a Goldstone mode corresponding to a spontaneously broken $U(1)$ symmetry. In this article we map the Kawasaki–Ising model for a classical lattice gas into a quantum model for a superfluid and establish a connection between the normal density fluctuations of the first and the Goldstone mode of the second. The fact that the quantum model we obtain describes a superfluid derives from an inequality by Penrose and Onsager which gives a lower bound to the Bose–Einstein condensate density. Mathematically, the Goldstone mode can be described by means of a "quantum" extension of the local algebra of the Ising model. The classification of its irreducible representations requires an additional $U(1)$ phase factor and the corresponding $U(1)$ gauge symmetry is spontaneously broken for all finite values of the temperature and of the density.

KEY WORDS: Ising model; Monte Carlo dynamics; spontaneous gauge symmetry breaking.

1. THE PENROSE–ONSAGER INEQUALITY AND THE QUANTUM PHASES OF THE ISING MODEL

The problem of computing the long-time asymptotics of the various Monte Carlo dynamics for the Ising model is attracting much attention and several rigorous techniques have been developed (see, for instance, refs. 1–3). Two popular algorithms are the Glauber algorithm, in which one flips one spin at a time, and the Kawasaki algorithm, in which the only process which is allowed is the exchange of two antiparallel spins on the same bond. The Kawasaki algorithm conserves the total magnetization or—in the language of the lattice gas interpretation of the Ising model—the total

[1] School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540.
[2] Present address: Department of Mathematics, University of Toronto, 3359 Mississagua Road North, Mississagua, Ontario, L5L 1C6, Canada.

822/77/1-2-6

0022-4715/94/1000-0077$07.00/0

particle number. These Monte Carlo algorithms satisfy the detailed balance condition and, as a consequence, the corresponding generators can be conjugated to self-adjoint operators. Such operators can be interpreted as quantum mechanical Hamiltonians and some of the problems concerning the Monte Carlo dynamics can be reinterpreted as problems of quantum many-body theory. In this and in a companion article[8] we describe two different ways of readapting concepts and techniques originally developed for quantum mechanical problems to the realm of stochastic Ising models. In this article we show how the notions of Bose–Einstein condensation, of spontaneous breakdown of particle number symmetry, and of Goldstone mode shed new light on the slow decay of time-dependent correlation functions for the Kawasaki dynamics and provide useful nonperturbative tools for the study of this problem. In the second article, we show how dynamic cluster expansions previously developed in the context of the theory of quantum spin systems can be used to compute analytically the long-time asymptotics of the Glauber dynamics at high temperature.

Let us introduce some notation. We consider the ferromagnetic Ising model with magnetic field. The Hamiltonian is

$$H(\sigma) = - \sum_{\langle xy \rangle} \sigma_x \sigma_y - h \sum_x \sigma_x \tag{1.1}$$

and is restricted on a large cube $\Lambda \subset \mathbb{Z}^d$ with periodic boundary conditions. A spin configuration is a map $\sigma\colon \Lambda \to \{0, 1\}$ and the associated Gibbs measure is

$$\mu_G(\sigma) = \frac{1}{Z_\Lambda(\beta)} \exp[-\beta H(\sigma)], \quad \text{where } Z_\Lambda(\beta) = \sum_\sigma \exp[-\beta H(\sigma)] \tag{1.2}$$

If $\mathcal{O}$ is a classical observable of the form $\mathcal{O}(\sigma) = \prod_{x \in \operatorname{supp} \mathcal{O}} \sigma_x$, its expectation is

$$\langle \mathcal{O} \rangle_\Lambda = \frac{1}{Z_\Lambda(\beta)} \sum_\sigma \mathcal{O}(\sigma) \exp[-\beta H(\sigma)] \tag{1.3}$$

The Glauber dynamics is a stochastic process on the configuration space $\{0, 1\}^\Lambda$ having $\mu_G(\sigma)$ as equilibrium distribution and is defined as follows: given a configuration σ, one picks a site $x \in \Lambda$ at random and one flips the spin in x, obtaining the configuration σ^x with a probability rate

$$p(\sigma \to \sigma^x; \beta) = A(1 + \exp\{-\beta[H(\sigma^x) - H(\sigma)]\}) \tag{1.4}$$

where A is a normalization constant. This choice satisfies the detailed balance condition

$$\frac{p(\sigma \to \sigma^x; \beta)}{p(\sigma^x \to \sigma; \beta)} = e^{-\beta[H(\sigma^x) - H(\sigma)]} \tag{1.5}$$

The Kawasaki dynamics is defined in a similar way, except that the only process which is allowed is the exchange of two neighboring spins in case they are different, the probability $p(\sigma \to \sigma^{xy})$ of such a process being proportional to $(1 + \exp\{-\beta[H(\sigma^{xy}) - H(\sigma)]\})$. The equivalence with a quantum mechanical problem can be established by observing that thanks to the detailed balance condition in (1.5), the generator $\mathscr{L}$ of any of these Monte Carlo Dynamics is such that

$$\exp[-\beta H(\sigma)]\, \mathscr{L}(\sigma, \sigma') = \mathscr{L}^{+}(\sigma, \sigma') \exp[-\beta H(\sigma')] \qquad (1.6)$$

for all σ, σ'. Hence, the operator $\tilde{\mathscr{L}}$ whose matrix elements are

$$\tilde{\mathscr{L}}(\sigma, \sigma') \equiv e^{(\beta/2)\, H(\sigma)} \mathscr{L}(\sigma, \sigma')\, e^{-(\beta/2)\, H(\sigma')} \qquad (1.7)$$

is self-adjoint and can be interpreted as a quantum mechanical Hamiltonian.

The main observation of this article is that the quantum mechanical interpretation of the Kawasaki–Ising model leads to a quantum liquid made up of lattice particles with a hard-core repulsion and obeying Bose statistics, in the sense that the corresponding wavefunction is constrained to be completely symmetric with respect to permutations. The quantum mechanical system we obtain is a superfluid. Superfluidity is a state of matter which is characterized by several remarkable properties, such as flow without viscosity and the existence of quantized vortices. Due to phase cancellations, the physics of the Kawasaki–Ising model is quite different, but the mathematical analogy remains. The theory of superfluidity is based on the concept of Bose–Einstein condensation, according to which the quantum state of zero momentum has a macroscopic occupation number. The superfluid associated to the Kawasaki–Ising model is mathematically simpler to analyze than real-life systems such as liquid helium. In fact, an approximation proposed by Feynmann[9] and widely used for numerical studies of superfluid Helium-4[10] turns out to be exact for our model. Feynman's approximation is based on a variational ansatz for the ground-state wavefunction. In our case, the ground state is given by what we call a "Gibbs wavefunction" and can be seen as the square root of the Gibbsian probability distribution. The simplifying feature is that this wavefunction is exactly of the form of Feynman's ansatz. Feynman's approximation attracted the attention of Penrose and Onsager, who wrote an article[4] in 1956 to show that this ansatz wavefunction has a Bose–Einstein condensate, meaning that the momentum distribution function has a delta-function singularity at zero momentum. The strength of the delta function is called *condensate density* and we denote it by ρ_0. An inequality in Penrose and Onsager's paper implies that $\rho_0 > 0$ for all finite values of the magnetic field and of the inverse temperature β. The proof is simple but conceptually deep, as the existence of a Bose–Einstein condensate implies that particle number symmetry is broken by the Gibbs wavefunction. In this article, we illustrate the notion of spontaneous breakdown of particle number symmetry in some detail and we do not assume that the reader is already acquainted with it. For further discussions, we refer to the books of Anderson[11] and Nozières and Pines.[15]

To understand Bose–Einstein condensation for the Ising model, it is necessary to introduce a quantum field theory formalism in which the commutative algebra $\mathscr{A}$ of classical observables is replaced by a larger noncommutative algebra $\bar{\mathscr{A}}$ generated by local field operators $\psi(x)$, $\psi^+(x)$. These are nonrelativistic fields corresponding to lattice particles with a hard-core repulsion. They obey Bose statistics in the sense that the corresponding wavefunction is constrained to be completely symmetric with respect to permutations. The Hilbert space spanned by symmetric wavefunctions can be identified with the tensor product space $\mathscr{H} = \bigotimes_{x \in \Lambda} \mathbb{C}^2_x$, where the fiber $\mathbb{C}^2_x$ is spanned by the states $|0\rangle_x$ and $|1\rangle_x$ corresponding to configurations in which the site $x \in \Lambda$ is either empty or occupied. If σ is a spin configuration, then $|\sigma\rangle \in \mathscr{H}(\Lambda)$ is the vector such that

$$|\sigma\rangle = \bigotimes_{x \in \Lambda} |\sigma_x\rangle_x \tag{1.8}$$

The field operators $\psi(x)$ and $\psi^+(x)$ are the adjoints of each other and act only on the fiber $\mathbb{C}^2_x$. We have

$$\psi(x)\,|\sigma\rangle = \delta_{\sigma_x 1}\,|\sigma^x\rangle \tag{1.9}$$

Introducing also the Pauli operators

$$\begin{aligned}
\sigma_x^{(1)} &= \psi(x) + \psi(x)^+ \\
\sigma_x^{(2)} &= i\psi(x) - i\psi^+(x) \\
\sigma_x^{(3)} &= \psi^+(x)\,\psi(x) - \psi(x)\,\psi^+(x)
\end{aligned} \tag{1.10}$$

the density operator

$$\rho(x) = \psi^+(x)\,\psi(x) \tag{1.11}$$

and the Ising operator

$$\mathbb{H} = -\sum_{\langle xy \rangle} \rho(x)\,\rho(y) + h \sum_x \rho(x) \tag{1.12}$$

we find that the Glauber and the Kawasaki generators are given by the following formulas:

$$\mathscr{L}_{\mathrm{G}} = \tfrac{1}{2} \sum_{x} (1 - \sigma_x^{(1)})\, e^{-\beta \mathbb{H}} (1 - \sigma_x^{(1)})\, e^{\beta \mathbb{H}} \tag{1.13}$$

$$\mathscr{L}_{\mathrm{K}} = \tfrac{1}{2} \sum_{\langle xy \rangle} (1 - \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y)\, e^{-\beta \mathbb{H}} (1 - \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y)\, e^{\beta \mathbb{H}} \tag{1.14}$$

respectively, where $\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y = \sum_{\alpha = 1, 2, 3} \sigma_x^{(\alpha)} \sigma_y^{(\alpha)}$. The Kawasaki generator is quite degenerate because it has a large number of equilibrium distributions; in fact, any restriction of the Gibbsian distribution to the set of spin configurations with a fixed total particle number is an equilibrium distribution by itself. To remove this degeneracy, one can either fix the total particle number or one can consider the operator

$$\mathscr{L}_{\mathrm{K}\varepsilon} = \mathscr{L}_{\mathrm{K}} + \varepsilon \mathscr{L}_{\mathrm{G}} \tag{1.15}$$

and evaluate the limit $\varepsilon \to 0$ *after* the infinite-volume limit has been taken. The first procedure is appropriate to the canonical ensemble and the second to the grand-canonical formalism. The quantum mechanical many-body Hamiltonian associated to the modified Kawasaki generator $\mathscr{L}_{\mathrm{K}\varepsilon}$ is

$$\tilde{\mathscr{L}}_{\mathrm{K}\varepsilon} = e^{\beta \mathbb{H}/2} \Bigg[\sum_{\langle xy \rangle} (1 - \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y)\, e^{-\beta \mathbb{H}} (1 - \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y) + \varepsilon \sum_{x} (1 - \sigma_x^{(1)})\, e^{-\beta \mathbb{H}} (1 - \sigma_x^{(1)}) \Bigg] e^{\beta \mathbb{H}/2} \tag{1.16}$$

and its (nondegenerate) ground state in the box Λ is the wavefunction

$$|\Psi_0\rangle_\Lambda = Z_\Lambda^{-1/2} \sum_{\sigma} e^{-(1/2)\beta H(\sigma)} |\sigma\rangle \tag{1.17}$$

If $\mathcal{O} = \psi^+(x_1) \cdots \psi^+(x_m)\, \psi(y_1) \cdots \psi(y_n)$ is an observable in $\mathscr{A}$, its infinite-volume expectation value in the ground state $|\Psi_0\rangle$ is

$$\langle \mathcal{O} \rangle_0 = \lim_{|\Lambda| \to \infty} \langle \Psi_0 |\, \mathcal{O}\, | \Psi_0 \rangle_\Lambda \tag{1.18}$$

An important observable which is mentioned above is the momentum distribution function

$$n(k) = \sum_{x} e^{ik \cdot x} \langle \psi^+(x)\, \psi(0) \rangle_0 \tag{1.19}$$

defined for $k \in [-\pi, \pi)^d$.

Theorem (Penrose, Onsager). For all finite values of β and of h we have that

$$\rho_0 \equiv \lim_{x \to \infty} \langle \psi^+(0)\, \psi(x) \rangle_0 \geqslant e^{-\beta h - 2d\beta} \rho^2 > 0 \tag{1.20}$$

where $\rho = \langle \rho(0) \rangle$ is the particle density. In particular, unless β is the inverse of the phase transition temperature, the momentum distribution function has the form $n(k) = \rho_0 \delta(k) + \tilde{n}(k)$, where $\tilde{n}(k)$ is an analytic function of k.

Proof. We have

$$\begin{aligned}
\langle \psi^+(x)\, \psi(0) \rangle_{0\Lambda} &= \langle \psi(x) \Psi_0 | \psi(0) \Psi_0 \rangle_\Lambda \\
&= Z_\Lambda^{-1} \sum_{\sigma : \sigma_0 = \sigma_x = 1} \exp\left\{ -\frac{\beta}{2} [H(\sigma^x) + H(\sigma^0)] \right\} \\
&= [\exp(-\beta h)] \left\langle \sigma_0 \sigma_x \exp\left[-\frac{\beta}{2} \left(\sum_{|y|=1} + \sum_{|y-x|=1} \right) \sigma_y \right] \right\rangle_\Lambda \\
&\geqslant [\exp(-\beta h - 2d\beta)] \langle \sigma_0 \sigma_x \rangle_\Lambda
\end{aligned} \tag{1.21}$$

Hence

$$\rho_0 \equiv \lim_{x \to \infty} \lim_{\Lambda \to \mathbb{Z}^d} \langle \psi^+(0)\, \psi(x) \rangle_{0\Lambda} \geqslant e^{-\beta h - 2d\beta} \rho^2 \tag{1.22}$$

A similar argument is used in ref. 4 to derive Eq. (35) in that paper.

The classification of the irreducible representations of the quantum algebra $\mathscr{A}$ gives rise to the notion of "quantum phase" for the Ising model which is finer than the classical notion of phase related to the classification of extremal infinite-volume Gibbs measures. Let us consider the action of the group $U(1)$ on the algebra $\mathscr{A}$ such that $\psi(x) \to e^{-i\phi} \psi(x)$. This $U(1)$ group is related to particle number symmetry because an observable is $U(1)$-invariant if and only if it corresponds to an operator that does not change the total particle number. The $U(1)$ symmetry is spontaneously broken in case there is a Bose–Einstein condensate. More precisely, phases which are pure in the quantum sense are associated to a phase factor $e^{i\phi} \in U(1)$ such that

$$\langle \psi(0) \rangle_\phi = \sqrt{\rho_0}\, e^{i\phi} \tag{1.23}$$

where $\langle \cdot \rangle_\phi$ denotes the expectation in such a phase. This follows from the fact that, in a pure phase, truncated correlation functions must decay and that, due to (1.20) and to the $U(1)$ invariance of the operator $\psi^+(x)\, \psi(0)$, we have

$$|\langle \psi(0) \rangle_\phi|^2 = \lim_{x \to \infty} \langle \psi^+(x)\, \psi(0) \rangle_\phi = \lim_{x \to \infty} \langle \psi^+(x)\, \psi(0) \rangle_0 = \rho_0 \tag{1.24}$$

independently of $\phi \in [0, 2\pi)$. Since the Penrose–Onsager inequality implies that $\rho_0 > 0$ for all finite values of β and h, we conclude that the $U(1)$ symmetry is broken for all these values.

To see this symmetry-breaking phenomenon in a more constructive way, let us introduce the states

$$|\Psi_\phi\rangle_\Lambda = Z_\Lambda^{-1/2} \sum_\sigma \exp\left[-\frac{1}{2}\beta H(\sigma) + i\phi \sum_x \sigma_x\right] |\sigma\rangle \tag{1.25}$$

These are not eigenstates of $\tilde{\mathscr{L}}_K$ in any cube Λ of finite volume, but nonetheless we have

$$\lim_{|\Lambda| \to \infty} |\langle \Psi_\phi | \Psi_{\phi'} \rangle_\Lambda| = \lim_{|\Lambda| \to \infty} Z_\Lambda^{-1} \sum_\sigma \exp\left[-\beta H(\sigma) + i(\phi - \phi') \sum_x \sigma_x\right] = \delta_{\phi\phi'} \tag{1.26}$$

and

$$\langle |\Psi_\phi | \tilde{\mathscr{L}}_{K\varepsilon} | \Psi_\phi \rangle_\Lambda | \leqslant \text{const} \cdot \varepsilon \, |e^{i\phi} - 1| \, |\Lambda| \tag{1.27}$$

Hence, asymptotically in the thermodynamic limit $|\Lambda| \to \infty$ and $\varepsilon \to 0$, the states $|\Psi_\phi\rangle$ become a family of degenerate, mutually orthogonal ground states of the Kawasaki Hamiltonian which are related by $U(1)$ gauge transformations.

Particle number symmetry is thus spontaneously broken in the Kawasaki–Ising model. Since $U(1)$ is a continuous Lie group, the Goldstone theorem applies[17] and the spectrum is gapless. Moreover, the nonperturbative techniques of quantum field theory which have been developed to illustrate and strengthen the Goldstone theorem are also available. These techniques include Ward identities for current–field correlation functions,[13] lower bounds on the spectral gap[14] (see also ref. 3), and various sum rules.[15, 16] Also the perturbation theory of models with a Goldstone mode relies on these nonperturbative results. In fact, since the spectrum of elementary excitations is gapless, perturbative expansions require systematic resummations and Ward identities play a crucial role, as they permit one to identify classes of diagrams which add up to zero and would not be possible to resum otherwise. See ref. 17 for an illustration of these ideas within the context of the BCS theory of superconductivity, which is another example of a quantum many-body theory with spontaneously broken particle number symmetry. Symmetries are always useful, also if hidden and/or spontaneously broken. Parisi and Sourlas[18] found that conservative Langevin equations for continuum analogs of the Kawasaki–Ising model have a hidden supersymmetry. The spontaneously broken $U(1)$ symmetry we discuss in this article is of a different nature, but the hope is that the formal connection between classical normal liquids and quantum superfluids can also prove useful. As an illustration of the quantum formalism we propose, in the next section we discuss three elementary Ward identities and prove a version of the Goldstone theorem in the Symanzik formulation.[7]

2. GOLDSTONE THEOREM AND WARD IDENTITIES

Let us start by introducing some notation. The current operator $\mathbf{j}(x)$ is defined as follows:

$$j_i(x) = \tfrac{1}{2}[\psi^+(x)\,\psi(x-e_i) - \text{h.c.}] + \tfrac{1}{2}e^{-\beta \mathbb{H}}[\psi^+(x)\,\psi(x-e_i) - \text{h.c.}]\,e^{\beta \mathbb{H}} \tag{2.1}$$

where h.c. stands for "Hermitian conjugate." We still denote by $\rho(x)$ the density operator in (1.11). The time-dependent versions of these operators are given by the following formulas in the original (nonsymmetric) representation for the Kawasaki–Ising model:

$$\rho(x,t) = e^{t\mathscr{L}_K}\rho(x)\,e^{-t\mathscr{L}_K}, \qquad \mathbf{j}(x,t) = e^{t\mathscr{L}_K}\mathbf{j}(x)\,e^{-t\mathscr{L}_K} \tag{2.2}$$

If $u(x)$ is a function on $\mathbb{Z}^d$, let

$$\nabla_i u(x) = u(x+e_i) - u(x) \tag{2.3}$$

be its gradient along the unit vector e_i in the direction of the ith coordinate axis. The continuity equation in operator form can be written as follows:

$$\frac{d}{dt}\rho(x,t) + \nabla_i j^i(x,t) = 0 \tag{2.4}$$

To verify the continuity equation (2.4), one has to compute a few commutators and make use of the commutation relation

$$[\rho(x), \psi^+(x)\,\psi(y) + \psi^+(y)\,\psi(x)] = \psi^+(x)\,\psi(y) - \psi^+(y)\,\psi(x) \tag{2.5}$$

The conserved current with respect to the self-adjoint version of the Kawasaki generator in (1.14) is

$$\mathbf{j}(x,t) = e^{(\beta/2)\mathbb{H}}\mathbf{j}(x,t)\,e^{-(\beta/2)\mathbb{H}} \tag{2.6}$$

Finally, if the modified generator $\tilde{\mathscr{L}}_{K\varepsilon}$ is used, then the conservation law for the current acquires an anomalous term and becomes

$$\frac{d\rho(x,t)}{dt} + \nabla_i j_i(x,t) = \varepsilon e^{(\beta/2)\mathbb{H}}[\psi(x) - \psi^+(x)]\, e^{-(\beta/2)\mathbb{H}} \tag{2.7}$$

At this point, all the relevant definitions are given and we can derive the Ward identities and the corresponding Goldstone theorems in Symanzik's form. Here, we discuss only an elementary example. Let us consider the correlation function $\langle T\jmath^\mu(x,t)\,\psi(0,0)\rangle_0$, where $\mu = 0, 1,..., d$ is a space-time index and we set $\jmath^0(x) = \rho(x)$. We adopt the Heisenberg picture so that

$$\langle T\jmath^\mu(z)\,\psi(0)\rangle_0 = \begin{cases} \langle \jmath^\mu(x)\, e^{-t\mathscr{L}_K}\psi(0)\rangle_0 & \text{if } \; t>0 \\ \langle \psi(0)\, e^{t\mathscr{L}_K}\jmath^\mu(x)\rangle_0 & \text{if } \; t<0 \end{cases} \tag{2.8}$$

where $z = (x, t)$. The expectation value $\langle\cdot\rangle_0$ is defined in (1.18). Using the continuity equation, we find the following Ward identity:

$$\begin{aligned} &\nabla_\mu\langle T\jmath^\mu(z)\,\psi(0)\rangle_0 \\ &\quad = \varepsilon[\langle Te^{(\beta/2)\mathbb{H}}\psi(z)e^{-(\beta/2)\mathbb{H}}\psi(0)\rangle_0 - \langle Te^{(\beta/2)\mathbb{H}}\psi^+(z)\, e^{-(\beta/2)\mathbb{H}}\psi(0)\rangle_0] \\ &\qquad + \delta(t)\langle[\jmath^0(x), \psi(0)]\rangle_0 \\ &\quad = \varepsilon[\tilde{G}(z) - G(z)] - \delta(x)\,\delta(t)\langle\psi(x)\rangle_0 \end{aligned} \tag{2.9}$$

where

$$\tilde{G}(z) = \langle Te^{(\beta/2)\mathbb{H}}\psi(z)\, e^{-(\beta/2)\mathbb{H}}\psi(0)\rangle_0 \tag{2.10}$$

and

$$G(z) = \langle Te^{(\beta/2)\mathbb{H}}\psi^+(z)\, e^{-(\beta/2)\mathbb{H}}\psi(0)\rangle_0 \tag{2.11}$$

Following Symanzik,[7] let us integrate this identity over a large cylinder $\mathscr{S}$ in space-time which contains the origin, and has the bottom basis on the plane $t = -T$ and the top on the plane $t = T$. By using (2.9) and integrating by parts, we find

$$\lim_{\varepsilon\to 0}\oint_{\partial S}\langle T\jmath^\mu(x,t)\,\psi(0,0)\rangle_0\, do_\mu = -\sqrt{\rho_0} \tag{2.12}$$

Hence, $\langle T\jmath^\mu(x,t)\,\psi(0,0)\rangle_0$ is not a summable function in $L^1(\mathbb{Z}^d\times\mathbb{R})$. Taking a Fourier transform of both members of (2.9), we find

$$i\omega\Gamma^0(\omega,k) + \sum_{i=1,...,d}(1-e^{ik_i})\,\Gamma^i(\omega,k) = \varepsilon[G(\omega,k) - \tilde{G}(\omega,k)] + \sqrt{\rho_0} \tag{2.13}$$

Here

$$\Gamma^{\mu}(\omega, k) = \int dt \sum_{x} e^{-i\omega t - ik \cdot x} \langle T\tilde{j}^{\mu}(x; t)\, \psi(0) \rangle_0 \tag{2.14}$$

$$G(\omega, k) = \int dt \sum_{x} e^{-i\omega t - ik \cdot x} G(t, x) \tag{2.15}$$

and

$$\tilde{G}(\omega, k) = \int dt \sum_{x} e^{-i\omega t - ik \cdot x} \tilde{G}(t, x) \tag{2.16}$$

Setting $\omega = 0$ and $k = 0$, we obtain a second Ward identity

$$G(0, 0) - \tilde{G}(0, 0) = \varepsilon^{-1} \sqrt{\rho_0} \tag{2.17}$$

which shows that the mass gap of the generator $\mathscr{L}_{G\varepsilon}$ tends to zero as $\varepsilon \to 0$.

Another important Ward identity is the following:

$$i\omega K_{0\nu}(\omega; k) + \sum_{i = 1,\ldots, d} (1 - e^{ik_i})\, K_{i\nu}(\omega; k) = 0 \tag{2.18}$$

where $K_{\mu\nu}(\omega; k)$ is the Fourier transform of the current–current correlation function $\langle Tj^{\mu}(x; t)\, j^{\mu}(0; 0)$. Setting $k_0 = \omega$, we find the following formula:

$$K_{\mu\nu}(\omega; k) = \left(\delta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{\omega^2 + k^2} \right) (K_0 + K_1 \omega^2 + k^2) + O_{\mu\nu}((\omega^2 + k^2)^2) \tag{2.19}$$

where we are assuming that the lack of isotropy of the lattice $\mathbb{Z}^d$ is reflected only in the error term $O_{\mu\nu}((\omega^2 + k^2)^2)$. In the case of BCS superconductors, the fact that $K_0 \neq 0$ is related to the Meissner effect.[19] Also in our case, we ought to have $K_0 \neq 0$. In fact, the density–density correlation function—which corresponds to the case $\mu = \nu = 0$—is not summable, as can be argued by the fact that the quantity

$$\lim_{\varepsilon \to 0} \sum_{x} \langle \rho(x)\, e^{-t\mathscr{L}_{K\varepsilon}} \rho(0) \rangle \tag{2.20}$$

is independent of t due to particle number conservation.[1]

ACKNOWLEDGMENTS

I have profited from a number of instructive and stimulating discussions about this work with several friends. Listing them according to the chronological order of the discussions, I would like to thank Horng Tzer

Yau, Jürg Fröhlich, Kristoph Gawedskij, Marco Isopi, Alain Sol Sznitman, Stefano Olla, Fabio Martinelli, Giovanni Jona Lasinio, Tom Spencer, Joel Lebowitz, and, most importantly, two helpful referees. This work was partially supported by the Ambrose Monell Foundation.

REFERENCES

1. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, Berlin, 1991).
2. A. De Masi and E. Presutti, *Mathematical Methods for Hydrodynamic Limits* (Springer-Verlag, Berlin, 1991).
3. S. L. Lu and H. T. Yau, Spectral gap and logarithmic Sobolev inequality of Kawasaki and Glauber dynamics, preprint (1993).
4. O. Penrose and L. Onsager, On the quantum mechanics of helium II, *Phys. Rev.* **104**:576 (1956).
5. L. Reatto, Bose–Einstein condensation for a class of wavefunctions, *Phys. Rev.* **183**:334 (1969).
6. D. Ruelle, Classical statistical mechanics of a system of particles, *Helv. Phys. Acta* **36**:183 (1963).
7. K. Symanzik, Euclidean proof of the Goldstone Theorem, *Commun. Math. Phys.* **6**:228 (1967).
8. C. Albanese and M. Isopi, Long time asymptotics of infinite particle systems, preprint (1994).
9. R. P. Feynman, *Phys. Rev.* **91**:1291 (1953).
10. K. Binder, *Monte Carlo Methods in Statistical Physics* (Berlin, Springer, 1979).
11. P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin-Cummings, London, 1984).
12. D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, 1978).
13. O. A. McBrian and T. Spencer, *Commun. Math. Phys.* **53**:299 (1977).
14. D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, New York, 1975).
15. P. Nozières and D. Pines, *The Theory of Quantum Liquids*, Vol. 2 (Benjamin, New York, 1966–1990).
16. J. Feldman, J. Magnen, V. Rivasseau, and E. Trubowitz, *Helv. Phys. Acta* **66** (1993).
17. J. Goldstone, *Nuovo Cimento* **19**:154 (1961).
18. G. Parisi and N. Sourlas, *Nucl. Phys. B* **206**:321 (1982).
19. J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964).